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Problems of fields on super Riemann surfaces

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Abstract. *Superconformal fields of type k on a compact super Riemann surface are defined and studied. It is shown that if there are non-trivial fermionic moduli, the space of such fields can behave in a highly anomalous way; with a module structure which depends strongly on the moduli. (This will only happen for $-1 \leq k \leq 2$, but this includes the case of superscalar fields.) The proofs are explicit for genus 1, the actual spaces of fields being computed; and implicit for higher genus, via a spectral sequence. The implications for superstring theory are indicated.*

1. INTRODUCTION

Let M be a super Riemann surface (SRS), of a compact type.

The version of superstring theory developed by Baranov and others [1] depends strongly on the theory of «superholomorphic vector bundles» on M. Specifically, the programme sets up super-form bundles ω^k for $k \in \mathbb{Z}$, and a super-Laplacian $\square_k$ on sections of ω^k. The translation of Mumford's isomorphism [2] then relates the (regularized) determinants of $\square_3$ and $5\square_1$ [3].

There are probably various abstract formulations of this theory which do what is claimed. My aim in this paper is to call attention to a problem which I have not so far seen discussed concerning the zero modes of the relevant operators. These are usually computed in a fairly ad hoc way which certainly works if M is a trivial («split», «canonical» or «without odd parameters» [4]) SRS. However, because the spaces of sections of ω^k (etc.) are modules over a Grassman algebra rather than a field, it is perfectly possible that the kernel and cokernel of $\square_k$ may not be direct summands, indeed not even free

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modules over the algebra. (They are finitely generated by ellipticity). This rather spoils any arguments based on naive counting of zero modes.

It is the aim of this paper to show that this unpleasant situation which is *theoretically* possible is in fact realized in a limited but important set of cases. Because the operator $\Box_k$ is rather complicated (it involves the metric) I shall restrict attention to the Dolbeault type operator $\bar{D}_k$ whose kernel is the module of superholomorphic sections of ω^k. In terms of supersmooth (G^∞) [5] sections $G^\infty(\omega^k)$, $\bar{D}_k$ maps

$$\bar{D}_k : G^\infty(\omega^k) \to G^\infty(\omega^k \otimes \bar{\omega}) \tag{1}$$

where $\bar{\omega}$ is the complex conjugate of ω. Our problem is to describe the kernel and cokernel of $\bar{D}_k$.

I shall exhibit the kind of complications which may arise with such a computation, using a filtration to simplify the work. It will follow from the filtration spectral sequence that the kernel and cokernel can indeed be simply described

a) when the SRS is of trivial type or

b) when the kernel *or* the cokernel vanishes, i.e. (for genus $g > 1$) when $k < -1$ or $k > 2$.

However for $k = -1, 0, 1, 2$ there is a *generic* non-zero differential which implies (e.g.) that the zero modes of $\bar{D}_k$ are not a direct summand in $G^\infty(\omega^k)$.

Since the spectral sequence argument is highly non-explicit and does *not* really give an idea of what the kernel and cokernel of $\bar{D}_k$ are, I shall first devote some time to the relatively simple calculation for $g = 1$, the super-torus.

It shoud be noted that these results also have bad consequences for the theory of the Picard variety (via Coker ($\bar{D}_0$) and the Jacobi variety (via ker ($\bar{D}_2$)). I hope to return to these problems another time.

Lastly, I shall indicate how the problems raised by these results might be avoided in the context of the superstring determinants.

2. THE FORM BUNDLES AND THE COHOMOLOGY

For the general theory of SRS's I refer to the works of Crane and Rabin [6] and Baranov and Schwarz [7]. To sum up; take B to be an infinite Grassmann algebra

$$B = \lim_{\overleftarrow{\ell}} \Lambda_{\mathbb{C}}(v_1 \ldots v_\ell) \tag{2}$$

with $B = B_0 \oplus B_1$ and $v_i \in B_1$; let $\epsilon : B \to \mathbb{C}$ be the augmentation. If B is not infinite, it well know that even smooth theory (e.g. G^L or H^L) will produce badly behaved modules of functions [5]. The surfaces we consider («de Witt»SRS's) are constructed by gluing together coarse open sets $\epsilon^{-1}(U) \subset B$ via mappings $\epsilon^{-1}(U) \xrightarrow{F} \epsilon^{-1}(V)$ which are

(i) G^∞

(ii) Superanalytic (co-ordinate expressions do not contain $\bar{z}$ or $\bar{\theta}$)

(iii) Superconformal: if $\mathcal{D}_U$ is the rank 1 subsheaf of the tangent sheaf $\tau(\epsilon^{-1}(U))$ generated by the operator $D_U = (\partial/\partial\theta + \theta \cdot \partial/\partial z)|_U$, then τF maps $\mathcal{D}_U$ (isomorphically) to $\mathcal{D}_V$. If follows from this as usual that $D_U = (D_U\theta_V) \cdot D_V$ or, in the more usual formulation

$$D = (D\tilde{\theta})\tilde{D} \tag{3}$$

This has the practical implication that we can write the new co-ordinates

$$\begin{aligned} \tilde{z} &= f(z) + \theta\psi(z)\sqrt{f'(z)} \\ \tilde{\theta} &= \psi(z) + \theta\sqrt{f'(z) + \psi(z)\psi'(z)} \end{aligned} \tag{4}$$

where f and ψ are analytic functions from U to B_0, B_1 respectively.

Over an SRS it is easy to define a holomorphic or antiholomorphic super vector bundle [8]; we shall only be interested in line bundles, whose fibre is B. Most particularly, following [3], we consider the bundless ω, $\bar{\omega}$. ω is the bundle defined by the distributions $\mathcal{D}_U$ in (iii) above; it is a rank 1, holomorphic odd sub-bundle of TM.A section of ω accordingly has local representatives linked by the formula

$$\tilde{\phi} = (\tilde{D}\theta)\phi \tag{5}$$

$\bar{\omega}$ is the complex conjugate of ω and its sections satisfy the conjugate of (5). Let L be a holomorphic line bundle over M, and let $G^\infty(L)$ be its sheaf of germs of G^∞ sections. Then there is a unique complex linear super-differential operator (the Dolbeault operator)

$$\bar{D}_L : G^\infty(L) \to G^\infty(L \otimes \bar{\omega}) \tag{6}$$

such that:

1. $\bar{D}_L(w) = 0$ iff w is a holomorphic section
2. $\bar{D}_L(fw) = (\bar{D}_L f)w + (-1)^{\deg f}(f.\bar{D}_L w)$

for f a G^∞ function, and w in $G^\infty(L)$.

In fact, in local coordinates for a neighbourhood $\epsilon^{-1}(U_\alpha)$, $\bar{D}_L$ is just the appropriate operator $\bar{D}$ on the coordinate representative w_α. If the representatives transform by $w_\alpha = A_\alpha^\beta w_\alpha$, with A_α^β holomorphic, the representatives of $\bar{D}_L(w)$ transform by

$$\bar{D}_L(w_\alpha) = \bar{D}|U_\alpha(w_\alpha) = A_\alpha^\beta \cdot (\bar{D}|U_\alpha(\bar{\theta}_\beta)) \cdot D|U_\beta(w_\beta)$$

using the formula (3); and by (5) this implies they fit together to give a section of $L \otimes \bar{\omega}$ as claimed.

LEMMA 1. (i) *The sheaves $G^\infty(L)$, etc., are acyclic.*

(ii) *Let the sheaf of superholomorphic sections $\mathcal{O}_s(L)$ be the kernel of $\bar{D}_L$. Then the sequence*

$$0 \to \mathcal{O}_s(L) \to G^\infty(L) \xrightarrow{\bar{D}_L} G^\infty(L \otimes \bar{\omega}) \to 0 \tag{7}$$

is exact.

Proof. For part (i) see [9]. Part (ii) reduces to showing that $\bar{D}_L$ is an epimorphism, which is a simple computation on the components. ∎

Lemma 1 implies that (7) is a resolution of $\mathcal{O}_s(L)$. We therefore can apply the functor Γ (global sections) and obtain

$$\Gamma(\mathcal{O}_s(L)) = \ker \Gamma(\bar{D}_L) = H^0(M, \mathcal{O}_s(L)) \tag{8}$$

Coker $\Gamma(\bar{D}_L) = H^1(M, \mathcal{O}_s(L))$ and of course $H^i(M, \mathcal{O}_s(L)) = 0$ for $i > 1$. It is these two modules which we want to calculate; particularly, $H^0(M, \mathcal{O}_s(L))$ is the module of (global) superholomorphic sections of L.

Following a standard practice we write $\mathcal{O}_s$ for $\mathcal{O}_s(L)$ when the bundle L is trivial.

3. THE GENERAL PROBLEM OF FUNCTIONS

Before approaching the calculation using the spectral sequence let us look in particular at $\Gamma(\mathcal{O}_s) = H^0(M, \mathcal{O}_s)$. An element of $\Gamma(\mathcal{O}_s)$ is a G^∞ function $F : M \to B$ such that $\bar{D}F = 0$. Writing F as a sum of components it is easy to deduce that, in local coordinates on U,

$$F = u(z) + \theta v(z) \tag{9}$$

where u, v are the Z-extensions of analytic functions from $\epsilon(U) \subset \mathbb{C}$ to B. To find the global conditions on F, we need to know how u, v transform. We have

$$\tilde{u}(\tilde{z}) + \tilde{\theta}\tilde{v}(\tilde{z}) = u(z) + \theta v(z), \tag{10}$$

where $(\tilde{z}, \tilde{\theta})$ are given by equations (4). Expanding (10) gives

$$\begin{aligned} &\tilde{u}(f(z)) + \tilde{u}'(f(z)) \cdot \theta\psi(z)\sqrt{f'(z)} + (\psi(z) + \theta\sqrt{f'(z) + \psi(z)\psi'(z)}) \times \\ &\times (\tilde{v}(f(z)) + \tilde{v}'(f(z)) \cdot \theta\psi(z)\sqrt{f'(z)} = u(z) + \theta v(z) \end{aligned} \tag{11}$$

Hence, equating components,

$$\begin{aligned} &\tilde{u}(f(z)) + \psi(z)\tilde{v}(f(z)) = u(z) \\ &\psi(z)\tilde{u}'(f(z))\sqrt{f'(z)} + \tilde{v}(f(z))\sqrt{f'(z) + \psi(z)\psi'(z)} = v(z) \end{aligned} \tag{12}$$

These equations are rather complicated.

One important family of solutions is the «constants»

$$u(z) = b \in B;\ v(z) = 0 \tag{13}$$

However, we cannot in general set $u(z)$ equal to zero because of the presence of v-dependent terms in the first equation (12). In one case we can; that is, the trivial one where all functions $\psi(z)$ («odd parameters») can be taken equal to 0. Then we obtain the well-known result:

PROPOSITION. 1 *If M is a trivial SRS, the global sections of $\mathcal{O}_s$ are given by pairs (u, v) where*

(i) u *is a constant function* $M \to B$

(ii) v *is the z-extension of a holomorphic spinor (section of the square root of the canonical bundle) with values in* B ∎

In other words, in this case

$$H^0(M,\mathcal{O}_s) \cong (H^0(\epsilon M,\mathcal{O}) \oplus H^0(\epsilon M,\mathcal{O}(\sqrt{K}))) \otimes B \tag{14}$$

where $\sqrt{K}$ is the square root of the canonical bundle corresponding to the spin structure of M. This is the model of the result we would like to have. (Although it should be noted that the dimension of $H^0(\epsilon M,\mathcal{O}(\sqrt{K}))$ or «space of harmonic spinors» [10] is not locally constant for $g > 2$, as a function on the spinor Teichmüller space.) And, following some other results in SRS theory, we might hope that the equations (12) could be straightened out via filtration to give a twisted version of Proposition 1 in the general case.

This is not so. We shall give the proof that it is not via filtration theory in sections 4, 5. Meanwhile, let us examine the case of a super-torus M, where the problem can be simply described.

If the spin structure on M is non-trivial, there are no odd parameters and no problems. However, if M has a trivial spin structure, it may have an odd parameter β. It can then be written as the quotient of the «super-plane» B by the group Γ, free abelian on y_1, y_2, where [11]

$$\begin{aligned} &y_1(z,\theta) = (z+1,\theta) \\ &y_2(z,\theta) = (z+a+\theta\beta,\beta+\theta)\ (a \in B_0, \mathrm{Im}\,\epsilon(a) \neq 0,\ \beta \in B_1) \end{aligned} \tag{15}$$

A superholomorphic function on M is given by any superholomorphic function $F: B \to B$ which is invariant under y_1, y_2. If F is given by (9), we find

$$\begin{aligned} &u(z+1) = u(z),\ v(z+1) = v(z) \\ (16)\qquad &u(z+a) + \beta v(z+a) = u(z), \\ &\beta u'(z+a) + v(z+a) = v(z) \end{aligned}$$

Use the first equations to expand u, v in Fourier series:

$$(17)\qquad u(z) = \sum a_n e^{inz},\ v(z) = \sum b_n e^{inz} (a_n, b_n \in B)$$

We find from the second equations

$$(18)\qquad \begin{aligned} e^{ina}(a_n + \beta b_n) &= a_n \\ e^{ina}(n\beta a_n + b_n) &= b_n \end{aligned}$$

From the first equation, if $n \neq 0$ β divides a_n, hence from the second equation, $b_n = 0$ (since $\beta^2 = 0$). Substituting back, $a_n = 0$. On the other hand, if $n = 0$, the second equation gives nothing, while the first gives as a necessary and sufficient condition

$$(19)\qquad \beta b_0 = 0.$$

We find that $u(z) = a_0$ is a constant, as we'd expect, but that $v(z) = b_0$ is not just a constant (section of the spin bundle, which is here trivial), but one which is annihilated by the odd parameter β. Let $\mathrm{Ann}(\beta)$ be the ideal of annihilators of β in B. $\mathrm{Ann}(\beta)$ contains $\beta \cdot B$ but may be larger. It may, in particular, not be a free B-module.

PROPOSITION 2. *If M is the supertorus which is the quotient of B by the group Γ defined by (15), then $H^0(M, \mathcal{O}_s)$ is the set of functions independent of z :*

$$(20)\qquad \{a_0 + \theta b_0;\ a_0 \in B,\ b_0 \in \mathrm{Ann}(\beta)\}. \qquad \blacksquare$$

Of course if $\beta = 0$ (but only then!), b_0 is completely unrestricted. Notice that ω is here a trivial bundle and hence all ω^k's are; so proposition 2 has told us $H^0(M, \mathcal{O}_s(\omega^k))$ for all k. If we had Serre duality in its ordinary form [12] this would also tell us H^1. Unfortunately this does not work; but we can get round the problem as follows. Describe $H^0(M, \mathcal{O}_s)$ as the cohomology

$$H^1(\Gamma, \mathcal{O}_s(B))$$

of the group Γ acting on the (global) super-analytic functions (u, v) on the superplane B via (15). This can now be calculated in stages. The first generator y_1 simply reduces the problem to calculating the cohomology of $\mathbb{Z} \cdot y_2$ on the y_1-invariants, i.e. the Fourier series (17). And the formulae (18) show that H^1, like H^0, is trivial on all Fourier components with $n \neq 0$. A straightforward calculation on the 0-component gives

PROPOSITION 3. *If M is as in proposition 2, then $H^1(M, \mathcal{O}_s)$ is isomorphic to the direct sum $B/\beta \cdot B \oplus B$. The first summand corresponds to cocycles with no θ component, the second to those involving θ.* ∎

4. THE FILTRATION METHOD

We now attempt to calculate H^0, H^1 step by step in the general case, using the natural filtration of the sheaf $\mathcal{O}_s(\omega^k)$. This is defined as follows: if $\bar{B}$ is the augmentation ideal of B and $\bar{B}^{(p)}$ its pth power, then $\mathcal{O}_s(\omega^k)^{(p)}$ consists of those sections whose expression in local coordinates involves only z, θ and constants in $\bar{B}^{(p)}$. Clearly $\mathcal{O}_s(\omega^k)^{(p)} \supset \mathcal{O}_s(\omega^k)^{(p+1)}$; and our choice for B implies

$$\mathcal{O}_s(\omega^k) = \varprojlim_{p} \mathcal{O}_s(\omega^k)/\mathcal{O}_s(\omega^k)^{(p)} \tag{21}$$

We begin by finding the filtration quotients, and hope to use them to build up the cohomology of $\mathcal{O}_s(\omega^k)$ -first in finite stages, then passing to the limit via (21). As in proposition 1, let ϵM denote the body of M and $\sqrt{K}$ its spin bundle. Our first claim is

PROPOSITION 4. *For all $p > 0$, the sheaf $\mathcal{O}_s(\omega^k)^{(p)}/\mathcal{O}_s(\omega^k)^{(p+1)}$ is isomorphic to the direct sum*

$$(\mathcal{O}(\sqrt{K^k}) \oplus \mathcal{O}(\sqrt{K^{k+1}})) \oplus \bar{B}^{(p)}/\bar{B}^{(p+1)} \tag{22}$$

Proof. Suppose to simplify first that $k = 0$; a section of $\mathcal{O}_s$ is given locally by functions u, v transforming by (12). If we now suppose u, v have their coefficients in $\bar{B}^{(n)}$, the equations (12) become trivial mod $\bar{B}^{(n+1)}$, since $\psi(z)$ is odd and so is in $\bar{B}$. Hence

$$\begin{aligned} \tilde{u}(f(z)) &\equiv u(z) \bmod \bar{B}^{(n+1)} \\ \tilde{v}(f(z))\sqrt{f'(z)} &\equiv v(z) \bmod \bar{B}^{(n+1)} \end{aligned} \tag{23}$$

which gives the sum (22) in this case. For the general k, we need to replace equation (10) by the appropriate transformation equation

$$(24) \qquad (D\tilde{\theta})^k(\tilde{u} + \tilde{\theta}\tilde{v}) = (u + \theta v)$$

(cf (5)). Once again, all that survives of $(D\tilde{\theta})^k$ after $\bar{B}^{(n+1)}$ is eliminated is a $(\sqrt{f'(z)})^k$ and a similar analysis to the one above shows that the u 's transform like sections of $\sqrt{K^k}$ mod $\bar{B}^{(n+1)}$, while the v 's transform like sections of $\sqrt{K^{k+1}}$. ■

We now introduce the spectral sequence. This can be done in a number of ways: the most «classical» is to filter the complex for computing $H^*(\mathcal{O}_s(\omega^k))$. We shall do this, but we shall use the Čech cochain complex [13] to work out the cohomology. This is because our final result on obstructions is best expressed using the transition functions ψ of (4), which have the nature of Čech cocycles. We shall be brief with the details. Let $\mathcal{U} = \{U_\alpha, \alpha \in A\}$ be a finite covering of M by de Witt open sets which are such that any intersection of U'_α is contractible. (This can easily be achieved e.g. simplicially.) We shall write the Čech cochain complex

$$(25) \qquad C^*(k) = C^*(\mathcal{U}, \mathcal{O}_s(\omega^k)).$$

In particular, a 0-cochain c assigns to each α an element c_α in $\Gamma(U_\alpha, \mathcal{O}_s(\omega^k))$; a 1-cochain assigns to a pair (α, β) with $U_\alpha \cap U_\beta \neq \emptyset$ an element $c_{\alpha\beta}$ in $\Gamma(U_\alpha \cap \cap U_\beta, \mathcal{O}_s(\omega^k))$, and so on.

We have the standard result [14] that there is a canonical isomorphism

$$(26) \qquad H^*(\mathcal{U}, \mathcal{O}_s(w^k)) \xrightarrow{\ \cong\ } H^*(M, \mathcal{O}_s(\omega^k)).$$

We can therefore identify the two. The filtration on $\mathcal{O}_s(\omega^k)$ defined above gives rise to a filtration

$$(27) \qquad C^*(k)^{(p)} = C^*(\mathcal{U}, \mathcal{O}_s(\omega^k)^{(p)})$$

on the Čech cochains; and this in turn gives rise to a spectral sequence [15] $\{E_r, d_r\}$ with

$$(28) \qquad \begin{aligned} E_1^{pq} &= H^{p+q}(C^*(k)^{(p)}/C^*(k)^{(p+1)}) \\ E_\infty^{p,q} &= Gr^{p,q}(H^*(C^*(k))) = Gr^{p,q}(H^*(M, \mathcal{O}_s(\omega^k))). \end{aligned}$$

(The idea of a spectral sequence is to compute the homology of a filtered complex by «successive approximations». Each E_r is a complex with two gradings and differential $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$. Its homology is $H(E_r) = E_{r+1}$. d_r detects cochains which are cocycles up to order r but not up to order $r+1$.) We first note an important consequence of proposition 4:

PROPOSITION 5. *In the spectral sequence (28), the* E_1 *term is*

$$E_1^{pq} = (H^{p+q}(\epsilon M, \mathcal{O}(\sqrt{K^k})) \oplus H^{p+q}(\epsilon M, \mathcal{O}(\sqrt{K^{k+1}})) \otimes \bar{B}^{(p)}/\bar{B}^{(p+1)}. \tag{29}$$ ∎

Our rough approximation to the cohomology can therefore be seen more precisely as a description of the spectral sequence's starting point. This also tells us

COROLLARY. *If* $k < -1$ *or* $k > 2$, *the spectral sequence collapses and*

$$H^p(M, \mathcal{O}_s(\omega^k)) \cong [H^p(\epsilon M, \mathcal{O}(\sqrt{K^k})) \oplus H^p(\epsilon M, \mathcal{O}(\sqrt{K^{k+1}}))] \otimes B \tag{30}$$ ∎

In fact, if $k > -1$, $c_1(\sqrt{K^k})$ and $c_1(\sqrt{K^{k+1}})$ are both negative so $H^0(\epsilon M, \mathcal{O}(\sqrt{K^k}) \oplus \mathcal{O}(\sqrt{K^{k+1}}) = 0$. Conversely $H^1 = 0$ if $k > 2$, using Serre duality. But all differentials in the spectral sequence increase total degree by 1, so in both these cases they must be zero. The isomorphism (30) of B-modules is now easily constructed – though it may not be canonical.

Up till now we have not separated out the even and odd fields – and in general it is simpler not to. However, we should notice that all modules such as $\mathcal{O}(\sqrt{K^k})$ carry a $\mathbb{Z}_2$ – grading (even and odd), which carries over into exact sequences such as (7), into the cohomology, and into the spectral sequence. (For example, a field of form (9) is *even* if $u(z)$ has values in B_0 and $v(z)$ in B_1.) It follows that in the expression (29) for $E_1^{p,q}$, the component

$$H^{p+q}(\epsilon M, \mathcal{O}(\sqrt{K^{k+\lambda}}) \otimes \bar{B}^{(p)}/\bar{B}^{(p+1)} \quad (\lambda = 0, 1) \tag{31}$$

has the parity of $p + \lambda$. We shall use this in the next section.

5. THE FIRST DIFFERENTIAL

We shall now show the kind of problems encountered in computing $H^*(M, \mathcal{O}_s(\sqrt{K^k}))$ for the «critical values» of k, by decribing the first differential d_1 in the spectral sequence (28). This should be seen as the obstruction to solving the equations

(12) (or appropriate generalizations) to first order, given a zero order solution in the E_1 term (29). It is a homomorphism

(32) $$d_1 : E_1^{pq} \to E_1^{p+1,q}$$

which relative to the $\mathbb{Z}_2$ grading of (31) is parity reversing. In other words, we have for all p, two homomorphisms (the only non-zero ones)

(33)
$$\begin{aligned} d_1(0) &: H^0(\epsilon M, \mathcal{O}(\sqrt{K^k}) \otimes \bar{B}^{(p)}/\bar{B}^{(p+1)} \to \\ &\to H^1(\epsilon M, \mathcal{O}(\sqrt{K^{k+1}}) \otimes \bar{B}^{(p+1)}/\bar{B}^{(p+2)} \\ d_1(1) &: H^0(\epsilon M, \mathcal{O}(\sqrt{K^{k+1}})) \otimes \bar{B}^{(p)}/\bar{B}^{(p+1)} \to \\ &\to H^1(\epsilon M, \mathcal{O}(\sqrt{K^k}) \otimes \bar{B}^{(p+1)}/\bar{B}^{(p+2)}. \end{aligned}$$

We shall see later (Prop. 9) that $d_1(0)$ is zero, so at least to this order (maybe to all) the bosonic part is unobstructed. The most interesting calculations accordingly concern $d_1(1)$. We use the equations which correspond to (12) for the bundle ω^k. As these are complicated, we omit products $\psi\psi'$, which do not contribute to first order. Suppose then that $Z_\alpha = (z_\alpha, \theta_\alpha)$ is a coordinate on U_α, and let the overlap map from U_β to U_α, $Z_\alpha Z_\beta^{-1}$ be defined by the pair $(f_{\alpha\beta}, \psi_{\alpha\beta})$ as in (4). Then if our section of $\mathcal{O}_s(\omega^k)$ is given by local representatives $u_\alpha(z_\alpha) + \theta_\alpha v_\alpha(z_\alpha)$, we find that

(34) $$(u_\alpha(f_{\alpha\beta}(z_\beta)) + \psi_{\alpha\beta}(z_\beta)v_\alpha(f_{\alpha\beta}(z_\beta)))\ (f'_{\alpha\beta}(z_\beta))^{k/2} = u_\beta(z_\beta)$$

(35)
$$\begin{aligned} (\psi_{\alpha\beta}(z_\beta)u'_\alpha(f_{\alpha\beta}(z_\beta)) + v_\alpha(f_{\alpha\beta}(z_\beta)))\ (f'_{\alpha\beta}(z_\beta))^{k+1/2} \\ + k u_\alpha(f_{\alpha\beta}(z_\beta))\psi'_{\alpha\beta}(z_\beta)\ (f'_{\alpha\beta}(z_\beta))^{k-1/2} \\ = v_\beta(z_{\alpha\beta}). \end{aligned}$$

To find $d_1(1)$ in (33), we first suppose that we are given a section of $H^0(\epsilon M, \mathcal{O}_s(\sqrt{K^{k+1}})) \otimes \bar{B}^{(p)}$ defined by $\{v_\alpha(z_\alpha)\}$ so that

(36) $$v_\alpha(f_{\alpha\beta}(z_\beta))\ (f'_{\alpha\beta}(z_\beta))^{k+1/2} = v_\beta(z_\beta) \bmod \bar{B}^{(p+2)}$$

Then *any* $\{u_\alpha(z_\alpha)\}$ of filtration $p+1$ will satisfy equations (34) and (35) mod $\bar{B}^{(p+1)}$, since the ψ 's are of filtration ≥ 1. This is what is meant by our computation to zero order. The problem, accordingly, is whether we can find u_α 's to solve equation (34) mod $\bar{B}^{(p+2)}$ (eq. (35) is still no problem). This will be the case if the cocycle

(37) $$c_{\alpha\beta} = \psi_{\alpha\beta}(z_\beta)v_\alpha(f_{\alpha\beta})(z_\beta))\ (f'_{\alpha\beta}(z_\beta))^{k/2}$$

in $C^1(\mathcal{U}, \mathcal{O}(\sqrt{K^k})) \otimes \bar{B}^{(p+1)}/\bar{B}^{(p+2)}$ is a coboundary. The cohomology class $[c_{\alpha\beta}]$ is accordingly $d_1(1)$ of the element v defined by (36).

Using (35) we rewrite

$$c_{\alpha\beta} = \psi_{\alpha\beta}(z_\beta) v_\beta(z_\beta) \cdot f'_{\alpha\beta}(z_\beta)^{-\frac{1}{2}} \tag{38}$$

Now from the general super Teichmüller theory [13], we find that $\{\psi_{\alpha\beta}(z_\beta) f'_{\alpha\beta}(z_\beta)^{-\frac{1}{2}}\}$ defines a class $\hat{\psi}$ in $H^1(\mathcal{U}, \mathcal{O}(\sqrt{K^{-1}})) \otimes \bar{B}^{(1)}/\bar{B}^{(2)}$ which is the «leading term of the fermionic moduli». This is, as in (26), the cohomology of M, i.e. by Riemann-Roch [14],

$$H^1(\epsilon M, \sqrt{K^{-1}}) \otimes \bar{B}^{(1)}/\bar{B}^{(2)} = (2g-2) \cdot \bar{B}^{(1)}/\bar{B}^{(2)} \tag{39}$$

(Serre dual to the space of 3/2 – forms). Putting it all together, we have

PROPOSITION 6. *The first differential on the odd part, $d_1(1)$ is the mapping*

$$H^0(\epsilon M, \mathcal{O}(\sqrt{K^{k+1}})) \otimes \bar{B}^{(p)}/\bar{B}^{(p+1)} \to$$
$$\to H^1(\epsilon M, \mathcal{O}(\sqrt{K^k})) \otimes \bar{B}^{(p+1)}/\bar{B}^{(p+2)} : \quad d_1(1)(v) = v \cdot \hat{\psi}$$

where the product is a combination of (a) the cohomology product

$$\begin{aligned} H^0(\epsilon M, \mathcal{O}(\sqrt{K^{k+1}})) &\otimes H^1(\epsilon M, \mathcal{O}(\sqrt{K^{-1}})) \\ &\to H^1(\epsilon M, \mathcal{O}(\sqrt{K^k})) \end{aligned} \tag{40}$$

and (b) the product in GrB. ∎

As the Corollary to Prop. 5 indicates, this differential is not often non-zero. The main interest, of course, is in the pairing (40), which can only be non-zero when $k = -1, 0, 1, 2$. In many cases, there are no harmonic spinors and so the pairing is also zero for $K = 0, 1$. But our general result is that $d_1(1)$ is as non-trivial as it can be.

LEMMA 1. *Given line bundles L_1, L_2 over ϵM and a non-zero section $s \in H^0(\epsilon M, \mathcal{O}(L_1))$, the map*

$$H^1(\epsilon M, \mathcal{O}(L_2)) \to H^1(\epsilon M, \mathcal{O}(L_1 \otimes L_2))$$

defined by multiplication by s is onto.

Proof. Use the sheaf exact sequence

$$0 \to \mathcal{O}(L_2) \xrightarrow{s} \mathcal{O}(L_1 \otimes L_2) \to J \to 0 \tag{41}$$

where J is the «skyscraper sheaf» on the zeros of s [15]. The cohomology sequence of (41) proves the lemma, since $H^1(\epsilon M, J) = 0$. ∎

This result, while indicating the non-triviality of the differential, does not give much hold on its dependence on $\hat{\psi}$. In the extreme cases – which are not the interesting ones – the pairing (40) and hence the calculation are simple: for $k = -1$, (40) is just

$$\begin{aligned} &H^0(\epsilon M, \mathcal{O}) \otimes H^1(\epsilon M, \mathcal{O}(\sqrt{K^{-1}})) \\ &\qquad = \mathbb{C} \otimes H^1(\epsilon M, \mathcal{O}(\sqrt{K^{-1}})) \\ &\qquad \xrightarrow{\cong} H^1(\epsilon M, \mathcal{O}(\sqrt{K^{-1}})) \end{aligned} \tag{42}$$

while for $k = 2$, it is the Serre duality pairing

$$\begin{aligned} &H^0(\epsilon M, \mathcal{O}(\sqrt{K^3})) \otimes H^1(\epsilon M, \mathcal{O}(\sqrt{K^{-1}})) \to \\ &\qquad \to H^1(\epsilon M, \mathcal{O}(\sqrt{K^2})) = \mathbb{C} \end{aligned} \tag{43}$$

However, when $k = 0, 1$ the variability of the space $h^0 = H^0(\epsilon M, \mathcal{O}(\sqrt{k})) \cong H^1(\epsilon M, \mathcal{O}(\sqrt{k}))$ means that it is hard to find a simple description. If the spin structure is *even*, there will often be no harmonic spinors ($h^0 = 0$), and so $d_1(1) = 0$ automatically. But is the structure is *odd*, the simplest case (the only one if $g < 4$ [10]) is that dim $h^0 = 1$, and this is the best example of what can happen. In this case, if we take $k = 0$, lemma 1 implies that the pairing

$$\begin{aligned} &H^0(\epsilon M, \mathcal{O}(\sqrt{K})) \otimes H^1(\epsilon M, \mathcal{O}(\sqrt{K^{-1}})) = \\ &= h^0 \otimes H^1(\epsilon M, \mathcal{O}(\sqrt{K^{-1}})) \to H^1(\epsilon M, \mathcal{O}) \end{aligned} \tag{44}$$

is onto: $\mathbb{C}^{2g-2} \to \mathbb{C}^g$. (This is the first obstruction for the θ part of *scalar* fields). From this it is not hard to find an expression for exactly when $d_1(1)$ vanishes, see proposition 8 below. Correspondingly, if $k = 1$, we get a pairing

$$\begin{aligned} &H^0(\epsilon M, \mathcal{O}(K)) \otimes H^1(\epsilon M, \mathcal{O}(\sqrt{K^{-1}})) = \\ &= \mathbb{C}^g \otimes \mathbb{C}^{2g-2} \to h^0(\cong \mathbb{C}) \end{aligned} \tag{45}$$

which is the adjoint of (44) under Serre duality; and this is determining for the «half-forms». The pairing (45) is non-degenerate on $\mathbb{C}^g$.

Let use now try to go from the pairing (44) to the actual differential. Suppose given a spin structure $\sqrt{K}$ on ϵM with $h^0 \cong \mathbb{C}$. Let $s \in H^0(\epsilon M, \mathcal{O}(\sqrt{K}))$ be «the» non-vanishing section: choose bases $x_1, \ldots, x_{2g-2}$ for $H^1(\epsilon M, \mathcal{O}(\sqrt{K^{-1}}))$ and $y_1, \ldots, y_g$ for $H^1(\epsilon M, \mathcal{O})$ so that $s \cdot x_i = y_i (i \leq g)$ and $= 0\ (i > g)$. Now any element $\hat{\psi}$ in $H^1(\epsilon M, \mathcal{O}(\sqrt{K^{-1}})) \otimes \bar{B}^{(1)}/\bar{B}^{(2)}$ has a unique expression

$$\hat{\psi} = \sum_{i=1}^{2g-2} x_i \otimes \eta_i (\eta_i \in \bar{B}^{(1)}/\bar{B}^{(2)}) \tag{46}$$

while $v \in H^1(\epsilon M, \mathcal{O}(\sqrt{K}) \otimes \bar{B}^{(p)}/\bar{B}^{(p+1)}$ has form $v = s \otimes \zeta \quad (\zeta \epsilon \bar{B}^{(p)}/\bar{B}^{(p+1)})$ We can now state

PROPOSITION 8. *Let M be an SRS such that the corresponding spin – RS is of odd type with a single harmonic spinor s. Let the leading term $\hat{\psi}$ of the fermionic moduli of M be given by (46), and let x_i, y_i be as above. Then in the spectral sequence for $H^*(M, \mathcal{O}_s(\omega^0))$, $d_1(1)$ is given by*

$$d_1(1)(s \otimes \zeta) = \sum_{i=1}^{g} y_i \otimes \zeta \eta_i \tag{47}$$

■

COROLLARY. *The differential $d_1(1)$ is generically non-zero, under the above conditions.* In fact; for a generic M, some $\eta_i \neq 0 (1 \leq i \leq g)$; and for a generic ζ, ζ does not annihilate all η_i's since B is supposed infinite. ■

The parallel results to proposition 8 for $k = -1, 1, 2$ are similar, and will be left to the reader to formulate. The important fact to note is that in all cases the statement that $d_1(1)$ is generically non-trivial holds (for $k = 1$ we must suppose that $h_0 \neq 0$). So that the naive characterization of «fields of type k» as pairs (bosonic k-field, fermionic $(k+1)$ -field) [16] breaks down.

We now turn to the bosonic differential $d_1(0)$ of (33), and prove.

PROPOSITION 9. *$d_1(0)$ is identically zero in all cases.*

Proof. There are only two cases to consider, $k = 0, 1$; all the others are zero because one of the two groups is. To find $d_1(0)$ we suppose $u = \{u_\alpha\}$ given, and find from (35) the condition for $\{v_\alpha\}$ to exist mod $\bar{B}^{(p+2)}$. Suppose $k = 0$; then u is a section of $\mathcal{O}$, i.e. constant. So $u'_\alpha = 0$ for all α. Since k is also zero, the obstruction terms in (35) vanish.

Now suppose $k = 1$. Here $d_1(1)$ is derived from a pairing

$$H^0(\epsilon M, \mathcal{O}(\sqrt{K})) \otimes H^1(\epsilon M, \mathcal{O}(\sqrt{K^{-1}})) \to H^1(\epsilon M, \mathcal{O}(K)) \tag{48}$$

analogous to (40), but defined using derivatives:

$$s \otimes t \longrightarrow s' \cdot t + ks \cdot t' = s' \cdot t + s \cdot t' \tag{49}$$

Here the dashes indicate the appropriate differentiation for line bundles:

$$D : H^i(\epsilon M, \mathcal{O}(L) \to H^i(\epsilon M, \mathcal{O}(L \otimes K))$$

But the right hand side of (49) is simply the derivative of $s \cdot t \in H^i(\epsilon M, \mathcal{O})$. Now it is standard that any element of $H^i(\epsilon M, \mathcal{O})$ can be represented by a locally constant (flat) cocycle (cf [17]), and hence has zero derivative in $H^i(\epsilon M, \mathcal{O}(K))$. This completes the proof. ∎

6. DISCUSSION

The above results on the spectral sequence calculation of fields on M are partial, since we have no adequate description of later differentials, and they could be quite inaccessible. The above stopping point is a sensible one, in that we have seen that the differentials are non-trivial, so there is a problem; and we have some idea of their character. However, what we really would like is (as with the torus) to describe the space of k-fields in terms of the supermoduli of the surface. This represents quite a challenge, and the spectral sequence looks like a bad way to approach it.

To move to larger problems, how can we avoid using the spaces of fields or zero modes of the operator $\bar{D}_k$, $\square_k$ of §1? (The difficulties associated with $\bar{D}_k$ do continue to arise with $\square_k$). The answer may be that the objects of interest for physics are not the spaces of zero modes but the suitably regularized determinants (see [18]); and that the «regularizing» expressions, which involve objects like $Tr(\exp -t\square_k)$, should not give rise to the same problems.

However, we need to be careful at every step. For example, the operators $\bar{D}_k$ of (1) depend on the supermoduli of M, and so on the point in super Teichmüller space $ST(g)$ – say x – which M defines. In particular for $g = 1$ with the trivial spin structure, $\bar{D}_k(x)$ is a function of the parameters α, β of x as given in (15). If we now try to follow the definition of the determinant bundle over $ST(g)$ (see [19]), we find that the fibre should be

$$\begin{aligned} \det{}_x(\bar{D}_k) &= \Lambda(\ker \bar{D}_k(x))^* \underset{B}{\otimes} \Lambda\,(\operatorname{cok} \bar{D}_k(x)) \\ &= \Lambda\,(B \oplus \operatorname{Ann}(\beta))^* \underset{B}{\oplus} \Lambda(B/\beta \cdot B \oplus B) \end{aligned} \tag{50}$$

by propositions 2, 3. Here « $\wedge$ » is a «top B-exterior power» for finitely generated B-modules. Clearly the fact that the B-modules are no longer free makes it much more difficult to use (50) to define a line bundle over $ST(1)$. It *can* be done by using the exact sequence

$$0 \to B \oplus \mathrm{Ann}(\beta) \to B \oplus B \overset{(0,\beta)}{\to} B \oplus B \to B \oplus B/_{\beta \cdot B} \to 0 \tag{51}$$

in which the middle terms are free modules. By standard methods, (51) identifies the right hand side of (50) with $\wedge\,(B \oplus B)^{\star} \otimes \wedge\,(B \oplus B)$. But this is very ad hoc, and we need to know that the determinants can be defined in the general situation.

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